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# PRESIDENCY UNIVERSITY

Department of Mathematics

Ph.D. Thesis Synopsis

## Title of the Proposed Thesis:

# ON THE TOTAL VARIATION OF CURVES UNDER CHAOS AND POSITIVE TOPOLOGICAL ENTROPY


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# Contents

**1 Introduction and Motivation** **6**

**2 Some Preliminaries** **8**

**3 Total Variation of curves under Transitive Maps on the Real Line** **10**
3.0.1 Sensitive dependence and the growth rate of total variation . . . . . 10
3.0.2 Total variation of curves for maps on the real line . . . . . 11
3.1 Conclusion and Open Issues . . . . . 12

**4 Total Variation of curves under Devaney Chaotic Maps on Finite graphs** **13**
4.0.1 Some preliminary definitions . . . . . 13
4.0.2 Existence of horseshoe for maps on graphs . . . . . 14
4.0.3 Total variation of curves for maps on graphs . . . . . 14
4.1 Conclusions and Open issues . . . . . 15

**5 Total Variation of curves under Transitive Maps on the Warsaw Circle** **16**
5.0.1 Warsaw circle and some chaotic properties . . . . . 16
5.0.2 Total variation of curves for maps on the Warsaw circle . . . . . 17
5.1 Conclusions and Open issues . . . . . 17

# List of publications

## Published article:

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# Abstract

The primary focus of this research is to study the change of the total variation of a curve under the iterations of a map on various spaces. We prove that if $f : \mathbb{R} \to \mathbb{R}$ is a transitive map on the real line and $\gamma$ is a non-constant curve having finite total variation then the total variation of $f^n \circ \gamma$ grows to infinity exponentially. A similar result is also proved for transitive maps on the Warsaw circle. But for the case of graph maps, transitivity condition is not enough to have exponential growth rate. In this case, if we replace transitivity by Devaney chaoticness then we can obtain an exponential growth rate.

# Chapter 1

# Introduction and Motivation

Topological dynamical system is defined as a pair $(X, f)$, where $X$ represents a topological space, referred to as the phase space and $f : X \to X$ is a continuous map. For a point $x \in X$, it's orbit under $f$, denoted as $orb(x)$, is defined as the sequence $x$, $f(x)$, $f^2(x)$, ..., $f^n(x)$, ...; where $f^n = f \circ f \ldots \circ f$ ($n$ times). In the subsequent sections, we presume $X$ to be a metric space. The central emphasis in the study of dynamical systems is on understanding the asymptotic behavior, specifically how the system evolves under successive iterations of the map $f$.

In 1975, Li and Yorke were the first to coin the phrase "chaos" for a continuous self map on a compact interval of the real line. In 1989, another precise definition of "chaos" is given by Devaney.

There are several ways for measuring the complexity of a dynamical system, including the Liapunov exponent, topological entropy, and the notions of Hausdorff and box dimensions.

Topological entropy for a continuous self map on a compact topological space was introduced by Adler, Konheim, and McAndrew (see [1]), as an invariant under topological conjugacy. It gives an idea of complexity for the orbit structure of a dynamical system. So far many studies have been performed on this notion. An one to one correspondence between the Devaney chaos and positive topological entropy for the case of interval maps and graph maps can be found in [13] and [17] respectively.

There is some connections between the total variation of any curve and chaos on some specific spaces, even though they seem to be two different concepts altogether. M. Misiurewicz and W. Szlenk were the first to connect the growth rate of total variation of an interval map with topological entropy (see [15]). In fact, they proved that if $I$ is a compact interval of the real line and $f : I \to I$ is a piecewise monotone continuous map and the total variation of $f^n$ on $I$ grows exponentially with respect to $n$, then the map $f$ has positive topological entropy. That is, they connect an analytic property of a system to the

dynamical property of this system. Then from the work of Chen et al. (see [6]) we have that for a chaotic interval map $f$, the total variation, $\mathrm{Var}(f^n; J) \to \infty$ as $n \to \infty$, where $J \subset I$ is any subinterval which can be considered as the image of a non constant curve $\gamma$ in $I$. To be precise, they showed that if the interval map $f$ has sensitive dependence on initial conditions then $\mathrm{Var}(f^n; J) \to \infty$ as $n \to \infty$ for any closed subinterval $J$ of $I$. It may be noted that, any Devaney chaotic map has sensitive dependence on initial conditions. Among many other things, they also prove the following interesting result.

**Theorem 1.0.1** *[6] Let $f : I \to I$ be a continuous map which is piecewise monotone. Then the following conditions are equivalent:*

1. *$f$ has a periodic point which is not a power of 2;*
2. *$f$ has positive topological entropy;*
3. *The growth rate for the total variation of $f^n$ on $I$ is exponential.*

The above theorem ensures that if the growth rate of the total variation is exponential then the system exhibits complicated dynamical behaviour. From this viewpoint in the subsequent chapters we shall discuss under what conditions the growth rate of the total variation of a curve under a continuous map on some spaces becomes exponential.

# Chapter 2

# Some Preliminaries

The following study is an attempt to understand how the total variation of a curve changes under iterations of transitive maps on the real line and on the Warsaw circle. We will also study the same for a Devaney chaotic graph maps.

Prior to beginning anything, we first recall some basic definitions.

**Definition 2.0.1** *For a map $f : X \to X$, a point $x \in X$ is called a periodic point of $f$ if $f^n(x) = x$, for some $n \in \mathbb{N}$. The smallest such $n$ is called the period of $x$. The set of all periodic points of $f$ is denoted as $Per(f)$.*

As we concern about the Devaney chaos so we need to recall the definition of it ([7]).

**Definition 2.0.2** *Let $f : X \to X$ be a continuous map. Then we say that $f$ is chaotic in the sense of Devaney if,*

1. *$f$ is topologically transitive, this means that for any two non empty open sets $U$ and $V$ of $X$ there exists a natural number $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$;*
2. *the set of all periodic points of $f$ are dense in $X$;*
3. *$f$ has sensitive dependence on initial conditions, i.e. there exists a constant $\delta > 0$ such that for all $x_0 \in X$ and any neighbourhood $U$ of $x_0$, there exists a point $y \in U$ and a number $n \in \mathbb{N}$ such that $d(f^n(x_0), f^n(y)) > \delta$.*

In the definition of Devaney chaos conditions (1), (2) and (3) are not logically independent. Banks et al. (see [3]) established that if the space $X$ is infinite, then (1) and (2) together implies (3).

If $f^2$ is transitive on $X$ then the map $f$ is called bitransitive. Moreover, $f : X \to X$ is called totally transitive if $f^n$ is transitive for every integer $n \geq 1$. We say the map $f$ is mixing if for any pair of non empty open sets $U$ and $V$ we can find a positive integer $N$

such that $f^n(U) \cap V \neq \emptyset$, for every $n \geq N$. Observe that mixing as a property is stronger than transitivity.

Let us now recall the notion of topological entropy.

Assume that $(X, d)$ is a compact metric space. For a continuous function $f : X \to X$ and an integer $n \geq 0$, define a new metric $d_n$ as follows,

$$d_n(x, y) = \max_{0 \leq k \leq n-1} d(f^k(x), f^k(y))$$

where $f^0$ is the identity map. The metric $d_n$ measures the maximum distance between the first $(n-1)$ iterates of two distinct points. A subset $E$ of $X$ is called $(n, \epsilon)$-separated if for all $x, y \in E$ with $x \neq y$, $d_n(x, y) \geq \epsilon$. Let $S_n(f, \epsilon)$ be the maximal possible cardinality of an $(n, \epsilon)$-separated set. It can be shown that the exponential growth rate of the number $S_n(f, \epsilon)$ tends to a limit as $\epsilon \to 0+$ (see [4]).

**Definition 2.0.3** *Let $X$ be a compact metric space and $f : X \to X$ be a continuous map. Then the topological entropy of $f$ is denoted by $h_{\text{top}}(f)$ and is defined by the following quantity,*

$$h_{\text{top}}(f) = \lim_{\epsilon \to 0+} \limsup_{n \to \infty} \frac{\log S_n(f, \epsilon)}{n}.$$

A continuous function $\gamma : [0, 1] \to X$ is called a curve in $X$. The trace of the curve $\gamma$ is denoted by $\gamma([0, 1])$ and is defined as the set $\{\gamma(t) : t \in [0, 1]\}$.

**Definition 2.0.4** *Let $P = \{t_0, t_1, \ldots, t_m\}$, where $a = t_0 < t_1 < \ldots < t_m = b$ be a partition of $[a, b]$. Then the total variation of the curve $\gamma : [a, b] \to X$ is denoted by $\text{Var}(\gamma; [a, b])$ and is defined by*

$$\text{Var}(\gamma; [a, b]) = \sup_{\mathcal{P}} \sum_{i=1}^{m} d\big(\gamma(t_i), \gamma(t_{i-1})\big),$$

*where the supremum is taken over all the partitions $\mathcal{P}$ of $[a, b]$. We say that the map $\gamma$ has finite length on $[a, b]$ if $\text{Var}(\gamma; [a, b]) < \infty$.*

In this research work, we will mainly generalize the Main Theorem 1 of [6] on some other spaces which is true for compact intervals on the real line.

# Chapter 3

# Total Variation of curves under Transitive Maps on the Real Line

The focus in this chapter is on studying the total variation of curves under the iterations of transitive maps on the real line. We wish to understand the role of total variation in capturing the complexity of the shape of curves under transitive maps.

### 3.0.1 Sensitive dependence and the growth rate of total variation

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous transitive map and $\gamma : [0, 1] \to \mathbb{R}$ be a curve having finite total variation. In the work of [6], the authors proved the following theorem.

**Theorem 3.0.1** *[6] Let $I$ be a compact subinerval of $\mathbb{R}$ and let $f : I \to I$ be continuous. Assume that $f$ has sensitive dependence on initial conditions on $I$. Then*

$$\lim_{n\to\infty} \mathrm{Var}(f^n; J) = \infty.$$

*for every closed subinterval $J$ of $I$.*

However sensitive dependence on initial conditions alone will not give us the exponential growth rate of total variation of a curve under iterations of a continuous map on $\mathbb{R}$. The following example makes this point clear.

Let $c_1 = 1, c_n = \frac{(n+1)^2}{n^2}$ for $n > 1$. We shall recursively define disjoint intervals $I_n = [a_n, b_n]$ on the non negative real line so that $I_n$ is at the left of $I_{n+1}$ with

$$|I_n| = \prod_{i=1}^{n-1} c_i, \text{ for } n > 1.$$

Define, $I_1 = [0, 1]$ (i.e. $a_1 = 0$, $b_1 = 1$); $a_2 = 10$, $b_2 = 10 + c_1$ and $\frac{a_{n+1}-b_n}{a_n-b_{n-1}} = \frac{3}{2}$ when $n \geq 3$. Observe that $|I_n| = \frac{n^2}{4}$, if $n \geq 3$. Now define a piecewise linear bijection $g$ on $\mathbb{R}$ in the following manner. Let $g(I_n) = I_{n+1}$ (i.e. $g(a_n) = a_{n+1}$, $g(b_n) = b_{n+1}$) and $g$ is linear on $I_n$

with slope $c_n$. Now define $g(x) = 2x + 10$, if $x < 0$. Then it is clear that $g$ has sensitive dependence on initial conditions. Just take $\delta = 1$ as the sensitivity constant. But letting $\gamma : [0, 1] \to I_j$, a nonconstant curve of finite total variation for any $j$, we see that,

$$\limsup_{n\to\infty} \frac{\log \mathrm{Var}(g^n \circ \gamma; [0, 1])}{n} = 0.$$

Observe that this map $g$ is neither topologically transitive nor it has any periodic point of period greater than 1. It has a fixed point at $x = -10$, but any point on the left of the fixed point goes off to $-\infty$ and any point on the right to the fixed point flies off to $\infty$ under iterations of $g$.

With this example we can say that the sensitivity condition is not enough for having the exponential growth rate.

### 3.0.2 Total variation of curves for maps on the real line

Now we state our first theorem:

**Theorem 3.0.2** *Let $\gamma : [0, 1] \to \mathbb{R}$ be a nonconstant curve of finite total variation and $f : \mathbb{R} \to \mathbb{R}$ be a transitive map then $\mathrm{Var}(f^n \circ \gamma; [0, 1]) \to \infty$ as $n \to \infty$. Moreover, the exponential growth rate of the total variation is strictly positive i.e.*

$$\limsup_{n\to\infty} \frac{\log \mathrm{Var}(f^n \circ \gamma; [0, 1])}{n} > 0.$$

**Remark 3.0.1** *The converse of Theorem 3.0.2 is not true. Considering the map $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x$, we see that any nonconstant curve $\gamma$ in $\mathbb{R}$ has the exponential growth rate of total variation under the iterations of $f$ whereas $f$ is not topologically transitive on $\mathbb{R}$.*

**Remark 3.0.2** *Our next example illustrates the fact that strictly positive exponential growth rate of the total variation does not necessairly imply that the map is bitransitive. Let us consider the map $f\colon \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} -x, & \text{if } x \leq 0; \\ -\frac{1}{n+1}, & \text{if } x = n, \text{where } n \text{ is an even positive integer;} \\ -3n, & \text{if } x = n, \text{where } n \text{ is an odd positive integer} \end{cases}$$

*and $f$ is affine linear in $[n, n + 1]$ so that $f$ is continuous on $\mathbb{R}$. Then $f$ is transitive but it is not bitransitive, as $f^2\big([0, \infty)\big) \subset [0, \infty)$. But the total variation of any curve $\gamma$ grows exponentially under iterations of $f$.*

## 3.1 Conclusion and Open Issues

In this chapter, we generalize the Theorem 3.0.1 for the case of the real line which is a non compact space. Also we provide an instance where the sensitivity condition is not enough for having an exponential growth rate of a curve for the case of the real valued maps. Now we leave the following question open regarding the case for real valued maps.

Let $X$ be a metric space. Let us say that a continuous map $f : X \to X$ has property $\mathcal{T}$ if the total variation of any non constant curve $\gamma$ in $X$ grows exponentially under iterations of $f$.

- Does there exist a continuous map $f : \mathbb{R} \to \mathbb{R}$ having sensitive dependence on initial conditions and a dense set of periodic points that does not possess property $\mathcal{T}$?

# Chapter 4

# Total Variation of curves under Devaney Chaotic Maps on Finite graphs

In this chapter we give an account of the same kind of result as 3.0.2. This discussion will be divided into three parts: first we recall the definition of a graph and state an important theorem by Misiurewicz et al. ([14]), then we state our theorem and lastly discuss some of the open issues in this regard.

### 4.0.1 Some preliminary definitions

Let us now first fix the notion of a graph and the topology on it. By an edge $e$ we mean that there is a continuous map $\phi\colon [0,1] \to e$ such that $(0,1)$ is homeomorphic to its image under $\phi$. The set $\phi\big((0,1)\big)$ will be called the interior of $e$ and is denoted by $\mathrm{int}(e)$, while the points $\phi(0)$, $\phi(1)$ are called the vertices of $e$.

By a graph $G$, we mean that it is a compact, connected, Hausdorff space which is a finite union of edges and vertices. It is well known that any graph $G$ can be embedded in $\mathbb{R}^3$. Thus, we can consider the metric on the graph is induced by the Euclidean distance of $\mathbb{R}^3$.

**Definition 4.0.1** *Let $G$ be a graph. Then an interval $I$ in the graph is a subset of $G$ such that it is homeomorphic to one of $[0,1]$, $[0,1)$, $(0,1)$ or $(0,1]$ and the endpoints can be atmost the vertices of the graph i.e., an interval is entirely contained in an edge of the graph.*

**Definition 4.0.2** *Let $f\colon G \to G$ be a continuous graph map. If there exist a closed interval $I$ and closed subintervals $J_1, J_2, \ldots, J_m$ of $I$ such that $f(J_i) = I$ for $i = 1, 2, \ldots, m$ and $\mathrm{int}(J_i) \cap \mathrm{int}(J_k) = \emptyset$ for $i \neq k$, then we say that $f$ has an $m$-horseshoe and we shall denote this horseshoe by $(I; J_1, J_2, \ldots, J_m)$. In this situation, we say that the image of $f$ on $I$ contains the set $I$ at least $m$ times.*

### 4.0.2 Existence of horseshoe for maps on graphs

Let $G$ be a graph and $\gamma\colon [0,1] \to G$ be a nonconstant curve having finite total variation. Now there is a fundamental result from the work of J. Llibre and M. Misiurewicz, namely Theorem B in [14]. This theorem guarantees that if the topological entropy of $f$ is positive then some iteration $f^m$ has a horseshoe inside a particular interval of the graph $G$. The existence of horseshoes is the crucial part for the exponential growth rate of a curve on the graph $G$. Let us now give the definition of horseshoe for a graph map. Considering the importance of this result we state it below.

**Theorem 4.0.1** *[14] If $f\colon G \to G$ is a continuous graph map and has topological entropy $h_{\text{top}}(f) > 0$, then there exist sequences $\{k_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$ of positive integers such that for each $n$, the map $f^{k_n}$ has an $s_n$-horseshoe and*

$$\limsup_{n\to\infty} \frac{\log s_n}{k_n} = h_{top}(f).$$

### 4.0.3 Total variation of curves for maps on graphs

Now we state our Theorem for the case of graph map:

**Theorem 4.0.2** *Let $G$ be a graph and $\gamma\colon [0,1] \to G$ be a non-constant curve of finite total variation and $f\colon G \to G$ be a Devaney chaotic map then $\operatorname{Var}(f^n \circ \gamma; [0,1]) \to \infty$ as $n \to \infty$ and the exponential growth rate of the total variation is strictly positive i.e.*

$$\limsup_{n\to\infty} \frac{\log \operatorname{Var}(f^n \circ \gamma; [0,1])}{n} > 0.$$

**Remark 4.0.1** *One thing to be noted that transitivity is not enough to have an exponential growth rate for the case of graph map. For example, consider the map $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ by $f(x) = x + \alpha \pmod 1$, where $\alpha$ is irrational. Then it is topologically transitive but neither $f$ has any periodic points nor it has sensitive dependence on initial conditions. As it is an isometry on the circle so any curve on the circle does not have the exponential growth rate.*

**Remark 4.0.2** *Exponential growth rate of the total variation is possible even when the map is not Devaney chaotic. The map $g(x) = 4x^3 - 3x$ on $[-1,1]$ is Devaney chaotic such that $g(-1) = -1$ and $g(1) = 1$. Consider $h : [-1,3] \to [-1,3]$ to be,*

$$h(x) = \begin{cases} g(x), & \text{if } x \in [-1,1]; \\ g(x-2) + 2\,, & \text{if } x \in [1,3]. \end{cases}$$

*This map is not topologically transitive as both $[-1,1]$ and $[1,3]$ are invariant under $h$. But the total variation of any non constant curve $\gamma$ will grow exponentially under iterations of $h$.*

## 4.1 Conclusions and Open issues

In this chapter, we generalize the Theorem 3.0.1 for the case of the graph maps which is a finite union of compact intervals of the real line or circles. Now we propose some open problems in this context:

- Under what conditions having property $\mathcal{T}$ implies the existence of a horseshoe for some higher iterates of a graph map?

- Let $f$ be a continuous self map on a graph $G$ having sensitive dependence on initial conditions. Does $f$ have property $\mathcal{T}$ ?

# Chapter 5

# Total Variation of curves under Transitive Maps on the Warsaw Circle

In this chapter, we focus on the case which is neither a compact interval of the real line nor a graph but is an one dimensional compact and connected subset of $\mathbb{R}^2$ known as the Warsaw circle, a quasi graph of degree two. Understanding the dynamics on this space contributes to the broader exploration of how topological structures influence the behavior of continuous maps under iterations. In the similar spirit of the previous chapters, we wish to investigate the growth rate of the total variation of a curve under the iterations of a transitive map on the Warsaw circle.

### 5.0.1 Warsaw circle and some chaotic properties

Warsaw circle is a simply connected but not a locally connected subspace of $\mathbb{R}^2$. We first recall the definition of the Warsaw circle and we denote it by $W$.

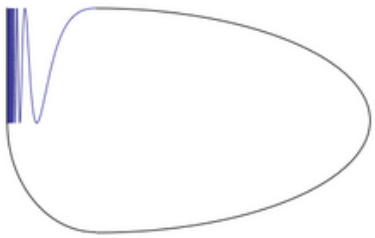

Figure 5.1: Warsaw circle

**Definition 5.0.1** *Let* $T = \{(t, \sin\frac{1}{t}) \in \mathbb{R}^2 : 0 < t \leq 1\}$. *Then* $\overline{T} = T \cup E$, *where* $E = \{(0, y) : -1 \leq y \leq 1\}$. *Take any simple arc* $C$ *joining* $(0, -1)$ *to* $(1, \sin 1)$. *Then,*

*$W = \overline{T} \cup C$ is called the Warsaw circle. Actually, the Warsaw circle is the union of the following five sets:*
$W_0 = \{(x, y) \in \mathbf{R}^2 : x = 0, -1 \leq y \leq 1\}$;
$W_1 = \{(x, y) \in \mathbf{R}^2 : x = 0, -3 \leq y \leq -1\}$;
$W_2 = \{(x, y) \in \mathbf{R}^2 : 0 \leq x \leq 1, y = -3\}$;
$W_3 = \{(x, y) \in \mathbf{R}^2 : x = 1, -3 \leq y \leq \sin 1\}$;
$W_4 = \{(x, y) \in \mathbf{R}^2 : 0 < x \leq 1, y = \sin \frac{1}{x}\}$.

There is an interesting result by Zhang et al. (see[19]) for maps on the Warsaw circle. We use this result to prove the growth rate of the total variation of a curve on the Warsaw circle. In view of the importance of this result, we restate it below.

**Theorem 5.0.1** *([19]) Let $f : W \to W$ be a continuous map on the Warsaw circle, then*

1. *$f$ is topological transitive if and only if $f$ is chaotic in the sense of Devaney;*
2. *$f$ is topological transitive if and only if $f$ is mixing; and*
3. *$f$ has horseshoe if $f$ is topological transitive.*

### 5.0.2 Total variation of curves for maps on the Warsaw circle

Now we state the Theorem for the case of Warsaw circle.

**Theorem 5.0.2** *Let $f\colon W \to W$ be a topologically transitive map and $\gamma\colon [0, 1] \to W$ be a non-constant curve of finite total variation then $\mathrm{Var}(f^n \circ \gamma; [0, 1]) \to \infty$ as $n \to \infty$ and the exponential growth rate of the total variation is strictly positive, i.e.*

$$\limsup_{n\to\infty} \frac{\log \mathrm{Var}(f^n \circ \gamma; [0, 1])}{n} > 0.$$

## 5.1 Conclusions and Open issues

In this last section we provide some open issues related to the dynamical behavior of maps on the Warsaw circle.

- Under what conditions having property $\mathcal{T}$ implies the existence of a horseshoe for some higher iterates of a map on the Warsaw circle ?
- Let $f : W \to W$ be a map on the Warsaw circle such that it has a periodic point of period 4. Does $f$ have property $\mathcal{T}$ ?
- Let $f : W \to W$ be a map on the Warsaw circle such that it has a periodic point which is not a power of two. Does $f$ have property $\mathcal{T}$ ?

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